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Research Article

Some Results on n -Times Integrated C -Regularized Semigroups

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We present a generation theorem of n -times integrated C -regularized semigroups and clarify the relation between differentiable $(n + 1)$ -times integrated C -regularized semigroups and singular n -times integrated C -regularized semigroups.

1. Introduction and Preliminaries

In 1987, Arendt [1] studied the n -times integrated semigroups, which are more general than C_0 semigroups (there exist many operators that generate n -times integrated semigroups but not C_0 semigroups).

In recent years, the n -times integrated C -regularized semigroups have received much attention because they can be used to deal with ill-posed abstract Cauchy problems and characterize the “weak” well-posedness of many important differential equations (cf., e.g., [2–18]).

Stimulated by the works in [2, 5–7, 9, 12–18], in this paper, we present a generation theorem of the n -times integrated C -regularized semigroups for the case that the domain of generator and the range of regularizing operator C are not necessarily dense, and prove that the subgenerator of an exponentially bounded, differentiable $(n + 1)$ -times integrated C -regularized semigroup is also a subgenerator of a singular n -times integrated C -regularized semigroup.

Throughout this paper, X is a Banach space; X^* denotes the dual space of X ; $L(X, X)$ denotes the space of all linear and bounded operators from X to X , it will be abbreviated to $L(X)$; $L(X)^*$ denotes the dual space of $L(X)$. By $C^1((0, +\infty), X)$ we denote the space of all continuously differentiable X -valued functions on $(0, +\infty)$. $C((0, +\infty), X)$ is the space of all continuous X -valued functions on $(0, +\infty)$.

All operators are linear. For a closed linear operator A , we write $D(A)$, $R(A)$, $\rho(A)$ for the domain, the range, the resolvent set of A in a Banach space X , respectively.

We denote by $A_0 = A|_{\overline{D(A)}}$ the part of A in $\overline{D(A)}$, that is,

$$D(A_0) := \{x \in D(A); Ax \in \overline{D(A)}\}, \quad A_0x = Ax, \text{ for } x \in D(A_0). \quad (1.1)$$

The C -resolvent set of A is defined as:

$$\rho_C(A) = \left\{ \lambda \geq 0; (\lambda - A) \text{ is injective, } R(C) \subset R(\lambda - A) \text{ and } (\lambda - A)^{-1}C \in L(X) \right\}. \quad (1.2)$$

We abbreviate n -times integrated C -regularized semigroup to n -times integrated C -semigroup.

Definition 1.1. Let n be a nonnegative integer. Then A is the subgenerator of an exponentially bounded n -times integrated C -semigroup $\{S(t)\}_{t \geq 0}$ if $(\omega, \infty) \subset \rho_C(A)$ for some $\omega \geq 0$ and there exists a strongly continuous family $S(\cdot) : [0, \infty) \rightarrow L(X)$ with $\|S(t)\| \leq Me^{\omega t}$ for some $M > 0$ such that

$$(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x \, dt \quad (\lambda > \omega, x \in X). \quad (1.3)$$

In this case, $\{S(t)\}_{t \geq 0}$ is called the exponentially bounded n -times integrated C -semigroup generated by $\tilde{A} := C^{-1}AC$.

If $C = I$ (resp., $n = 0$), then A is called a generator of an exponentially bounded n -times integrated semigroup (resp., C -semigroup).

We recall some properties of n -times integrated C -semigroup.

Lemma 1.2 (see [10, Lemma 3.2]). Assume that A is a subgenerator of an n -times integrated C -semigroup $\{S(t)\}_{t \geq 0}$. Then

- (i) $S(t)C = CS(t)$ ($t \geq 0$),
- (ii) $S(t)x \in D(A)$, and $AS(t)x = S(t)Ax$ ($t \geq 0, x \in D(A)$),
- (iii) $S(t)x = (t^n/n!)Cx + A \int_0^t S(s)x \, ds$ ($t \geq 0, x \in X$).

In particular, $S(0) = 0$.

Definition 1.3. Let $\omega \geq 0$. If $(\omega, \infty) \subset \rho_C(A)$ and there exists $\{S(t)\}_{t \geq 0} \subset L(X)$ such that

- (i) $S(0) = 0$ and $S(\cdot) : (0, \infty) \rightarrow L(X)$ is strongly continuous,
- (ii) for $\lambda > \omega$, $\int_0^\infty e^{-\lambda t} \|S(t)\| \, dt < \infty$,
- (iii) $(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x \, dt$, $\lambda > \omega, x \in X$,

then we say that $\{S(t)\}_{t \geq 0}$ is a singular n -times integrated C -semigroup with subgenerator A .

Remark 1.4. Clearly, an exponentially bounded n -times integrated C -semigroup is a singular n -times integrated C -semigroup. But the converse is not true.

2. The Main Results

Theorem 2.1. Let $M > 0$, $\omega \geq 0$ be constants, and let A be a closed operator satisfying $(\omega, \infty) \subset \rho_C(A)$. Assume that $\varphi(t)$ is the nonnegative measurable function on $[0, \infty)$. A necessary and sufficient condition for A is the subgenerator of an $(n+1)$ -times integrated C -semigroup $\{S(t)\}_{t \geq 0}$ satisfying

$$(A1) \limsup_{\lambda \rightarrow \infty} \|\lambda^{n+2} \int_0^\infty e^{-\lambda t} S(t) dt\| \leq M,$$

$$(A2) \|S(t) - S(s)\| \leq \int_t^s \varphi(u) e^{\omega u} du, \quad 0 \leq t \leq s, \text{ is that for } \lambda > \omega,$$

$$(i) \limsup_{\lambda \rightarrow \infty} \|\lambda(\lambda - A)^{-1} C\| \leq M,$$

$$(ii) \|[(\lambda - A)^{-1} C / \lambda^n]^{(m)}\| \leq \int_0^\infty e^{-(\lambda - \omega)t} t^m \varphi(t) dt, \quad m = 1, 2, \dots$$

Proof. Sufficiency. Let $\psi(t) = e^{\omega t} \varphi(t)$. Set

$$f(\lambda) = \int_0^\infty e^{-\lambda t} \psi(t) dt = \int_0^\infty e^{-(\lambda - \omega)t} \varphi(t) dt, \quad \lambda > \omega. \quad (2.1)$$

For $x^* \in X^*$, we have

$$\left| \left\langle \left[\frac{(\lambda - A)^{-1} C}{\lambda^n} x \right]^{(m)}, x^* \right\rangle \right| \leq \|x\| \cdot \|x^*\| \int_0^\infty e^{-\lambda t} t^m \varphi(t) dt$$

$$\leq \|x\| \cdot \|x^*\| \cdot f(\lambda)^{(m)}, \quad m = 1, 2, \dots \quad (2.2)$$

Using this fact together with Widder's classical theorem, it is not difficult to see that the existence of a measurable function $h(\cdot, x, x^*)$ with $|h(t, x, x^*)| \leq \|x^*\| \|x\| \varphi(t)$, a.e., ($t \geq 0$) such that

$$\left\langle \frac{(\lambda - A)^{-1} C}{\lambda^n} x, x^* \right\rangle = \int_0^\infty e^{-\lambda t} h(t, x, x^*) dt, \quad \lambda > \omega. \quad (2.3)$$

Let $H(t, x, x^*) = \int_0^t h(s, x, x^*) ds$, $t \geq 0$, $x^* \in X^*$. In view of the convolution theorem for Laplace transforms and from (2.3), we have

$$\left\langle \frac{(\lambda - A)^{-1} C}{\lambda^n} x, x^* \right\rangle = \lambda \int_0^\infty e^{-\lambda t} H(t, x, x^*) dt, \quad \lambda > \omega, \quad x^* \in X^*. \quad (2.4)$$

Using the uniqueness of Laplace transforms and the linearity of $h(\cdot, x, x^*)$ for each $x^* \in X^*$, $x \in X$, we can see that for each $t \geq 0$, $H(t, x, x^*)$ is linear and

$$|H(t+h, x, x^*) - H(t, x, x^*)| \leq \int_t^{t+h} |h(s, x, x^*)| ds \leq \|x\| \cdot \|x^*\| \int_t^{t+h} \varphi(s) ds. \quad (2.5)$$

Hence for all $t \geq 0$, there exists $S(t) \in L(X)^{**}$ such that

$$H(t, x, x^*) = \langle S(t)x, x^* \rangle, \quad x \in X, \quad x^* \in X^*, \quad (2.6)$$

$$\|S(t+h) - S(t)\| \leq \int_t^{t+h} \psi(s) ds, \quad t \geq 0, \quad h \geq 0, \quad (2.7)$$

$$\frac{(\lambda - A)^{-1}C}{\lambda^n} = \lambda \int_0^\infty e^{-\lambda t} S(t) dt. \quad (2.8)$$

Denote by $q : L(X)^{**} \rightarrow L(X)^{**}/L(X)$ the quotient mapping. Since $(\lambda - A)^{-1}C \in L(X)$, we deduce

$$0 = q\left(\frac{(\lambda - A)^{-1}C}{\lambda^n}\right) = \lambda \int_0^\infty e^{-\lambda t} q(S(t)) dt. \quad (2.9)$$

It follows from the uniqueness theorem for Laplace transforms that $q(S(t)) = 0$, that is, $S(t) \in L(X)$.

Combining (2.7) and (2.8) yields that $S(t) : [0, \infty) \rightarrow L(X)$ is strongly continuous and

$$\int_0^\infty e^{-\lambda t} \|S(t)\| dt \leq \int_0^\infty e^{-\lambda t} \int_0^t \psi(s) ds dt = \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} \psi(t) dt < \infty. \quad (2.10)$$

Now, we conclude that $\{S(t)\}_{t \geq 0}$ is an $(n+1)$ -times integrated C -semigroup satisfying (A2). Assertion (A1) is immediate, by (2.8) and (i).

Necessity. Let $\varphi(t) = e^{\omega t} \psi(t)$. Since $\{S(t)\}_{t \geq 0}$ is an $(n+1)$ -times integrated C -semigroup on X , we have

$$(\lambda - A)^{-1}C = \lambda^{n+1} \int_0^\infty e^{-\lambda t} S(t) dt \quad (2.11)$$

for $\lambda > \omega$. Noting that $\|S(t+h) - S(t)\| \leq \int_t^{t+h} \varphi(s) ds$ ($h \geq 0$) and $S(0) = 0$, we find

$$\|S(t)\| \leq \int_0^t \varphi(s) ds. \quad (2.12)$$

Then for any $y^* \in L(X)^*$ and $\lambda > \omega$, we obtain

$$\begin{aligned} \left\langle \frac{(\lambda - A)^{-1}C}{\lambda^n}, y^* \right\rangle &= \left\langle \lambda \int_0^\infty e^{-\lambda t} S(t) dt, y^* \right\rangle \\ &\leq \lambda \int_0^\infty e^{-\lambda t} \|S(t)\| \cdot \|y^*\| dt \leq \|y^*\| \int_0^\infty e^{-\lambda t} \varphi(t) dt. \end{aligned} \quad (2.13)$$

Therefore, there exists a measurable function $\eta(t)$ on $[0, \infty)$ with $|\eta(t)| \leq \psi(t)$ (a.e.) such that

$$\left\| \frac{(\lambda - A)^{-1}C}{\lambda^n} \right\| = \int_0^\infty e^{-\lambda t} \eta(t) dt. \quad (2.14)$$

Furthermore, by calculation, we have

$$\left\| \left[\frac{(\lambda - A)^{-1}C}{\lambda^n} \right]^{(m)} \right\| \leq \int_0^\infty e^{-\lambda t} t^m \psi(t) dt = \int_0^\infty e^{-(\lambda - \omega)t} t^m \varphi(t) dt, \quad m = 1, 2, \dots \quad (2.15)$$

Assertion (i) is an immediate consequence of (2.11) and (A1). \square

Remark 2.2. If $n = 0$ and $C = I$, then $\{S(t)\}_{t \geq 0}$ is an integrated semigroup in the sense of Bobrowski [2].

Theorem 2.3. Let $M > 0$, $\omega \geq 0$ be constants, and let A be a closed operator satisfying $(\omega, \infty) \subset \rho(A)$. Assume that A is a subgenerator of an $(n + 1)$ -times integrated C -semigroup $\{S(t)\}_{t \geq 0}$ and satisfies (ii) of Theorem 2.1 and $\limsup_{\lambda \rightarrow \infty} \|\lambda(\lambda - A)^{-1}\| \leq M$. If $A_0 = A|_{\overline{D(A)}}$ is a subgenerator of an n -times integrated C -semigroup $\{S_0(t)\}_{t \geq 0}$ on $\overline{D(A)}$, then for $\mu \in \rho(A)$, $x \in X$,

$$S(t)x = (\mu - A_0) \int_0^t S_0(s)(\mu - A)^{-1}x ds, \quad (2.16)$$

$$S(t)x = \lim_{\mu \rightarrow \infty} \mu \int_0^t S_0(s)(\mu - A)^{-1}x ds. \quad (2.17)$$

Proof. For $\mu \in \rho(A)$, $x \in X$, set $\{\hat{S}(t)\}_{t \geq 0}$ as follows:

$$\hat{S}(t)x = \mu \int_0^t S_0(s)(\mu - A)^{-1}x ds - S_0(t)(\mu - A)^{-1}x + \frac{t^n}{n!}(\mu - A)^{-1}Cx. \quad (2.18)$$

Since $S_0(t)$ is strongly continuous on $\overline{D(A)}$, $\hat{S}(t)$ is strongly continuous on X .

Fixing $\lambda > \omega$, we have

$$\begin{aligned} \lambda^{n+1} \int_0^\infty e^{-\lambda t} \hat{S}(t)x dt &= \lambda^n (\mu - \lambda) \int_0^\infty e^{-\lambda t} S_0(t)(\mu - A)^{-1}x dt + (\mu - A)^{-1}Cx \\ &= (\mu - \lambda)(\lambda - A)^{-1}C(\mu - A)^{-1}x + (\mu - A)^{-1}Cx \\ &= (\lambda - A)^{-1}Cx. \end{aligned} \quad (2.19)$$

It follows from the uniqueness of Laplace transforms that $S(t)x = \hat{S}(t)x$, $x \in X$. So we get (2.16). By the hypothesis $\limsup_{\lambda \rightarrow \infty} \|\lambda(\lambda - A)^{-1}\| \leq M$, we see

$$\begin{aligned}
S(t)x &= \lim_{\mu \rightarrow \infty} \left(\mu \int_0^t S_0(s)(\mu - A)^{-1}x \, ds - S_0(t)(\mu - A)^{-1}x + \frac{t^n}{n!}(\mu - A)^{-1}Cx \right) \\
&= \lim_{\mu \rightarrow \infty} \mu \int_0^t S_0(s)(\mu - A)^{-1}Cx \, ds,
\end{aligned} \tag{2.20}$$

and the proof is completed. \square

Now, we study the relation between differentiable $(n + 1)$ -times integrated C -semigroups and singular n -times integrated C -semigroups.

Theorem 2.4. *Let $\omega \geq 0$, and let A be a closed operator satisfying $(\omega, \infty) \subset \rho_C(A)$. Assume that $\varphi(t)$ is the nonnegative measurable function on $[0, \infty)$. The following two assertions are equivalent:*

- (1) *A is the subgenerator of a singular n -times integrated C -semigroup $\{U(t)\}_{t \geq 0}$ satisfying $\|U(t)\| \leq \varphi(t)e^{\omega t}$.*
- (2) *A is the subgenerator of an exponentially bounded $(n + 1)$ -times integrated C -semigroup $\{S(t)\}_{t \geq 0}$ satisfying*

$$\begin{aligned}
\|S(t) - S(s)\| &\leq \int_t^s \varphi(\tau)e^{\omega\tau} d\tau, \quad 0 \leq t \leq s, \\
S(t)x &\in C^1((0, +\infty), X), \quad \text{for } x \in X.
\end{aligned} \tag{2.21}$$

Proof. (1) \Rightarrow (2): we set

$$S(t)x := \int_0^t U(s)x \, ds, \quad t \geq 0. \tag{2.22}$$

Since $U(t)x$ is locally integrable on $[0, +\infty)$, $S(t)x$ is well-defined for any $x \in X$. It is easy to check that $S(t)x$ belongs to $C^1((0, +\infty), X)$.

For every $\lambda > \omega$, since

$$\|S(t)x\| = \left\| \int_0^t e^{-\lambda s} e^{\lambda s} U(s)x \, ds \right\| \leq e^{\lambda t} \int_0^t e^{-\lambda s} \|U(s)x\| \, ds \leq M e^{\lambda t} \|x\|, \tag{2.23}$$

we deduce that $S(t)$ is exponentially bounded.

Moreover, for $\lambda > \omega$, we have

$$\begin{aligned}
(\lambda - A)^{-1}Cx &= \lambda^n \int_0^\infty e^{-\lambda t} U(t)x \, dt = \lambda^{n+1} \int_0^\infty e^{-\lambda t} S(t)x \, dt, \\
\|S(t) - S(s)\| &= \left\| \int_t^s U(\tau) d\tau \right\| \leq \int_t^s \varphi(\tau)e^{\omega\tau} d\tau, \quad 0 \leq t \leq s.
\end{aligned} \tag{2.24}$$

Thus $\{S(t)\}_{t \geq 0}$ is the desired semigroup in (2).

(2) \Rightarrow (1): for any $x \in X$, we set

$$\begin{aligned} U(t)x &:= \frac{d}{dt}S(t)x, \quad \text{for } t > 0, \\ U(0)x &:= 0, \quad \text{for } t = 0. \end{aligned} \quad (2.25)$$

Then $U(t)x \in C((0, +\infty), X)$ and $U(0) = 0$.

Noting that

$$\|S(t+h) - S(t)\| \leq \int_t^{t+h} \varphi(s)e^{\omega s} ds, \quad (2.26)$$

we find

$$\left\| \frac{S(t+h) - S(t)}{h} \right\| \leq \frac{1}{h} \int_t^{t+h} \varphi(s)e^{\omega s} ds. \quad (2.27)$$

Since $S(t)x$ is continuously differentiable for $t > 0$, we get

$$\|U(t)\| \leq \varphi(t)e^{\omega t} \quad (\text{a.e.}). \quad (2.28)$$

Moreover, for $\lambda > \omega$, we have

$$\begin{aligned} \int_0^\infty e^{-\lambda t} \|U(t)\| dt &\leq \int_0^\infty e^{-(\lambda-\omega)t} \varphi(t) dt < \infty, \\ (\lambda - A)^{-1} Cx &= \lambda^{n+1} \int_0^\infty e^{-\lambda t} S(t)x dt = \lambda^n \int_0^\infty e^{-\lambda t} U(t)x dt. \end{aligned} \quad (2.29)$$

Thus, $\{U(t)\}_{t \geq 0}$ is a singular n -times integrated C -semigroup with subgenerator A . \square

Theorem 2.5. Let $M > 0$, $\omega \geq 0$ be constants, and let A be a closed operator satisfying $(\omega, \infty) \subset \rho(A)$. Let $\varphi(t)$ be the function in Theorem 2.4. If A is the subgenerator of a singular n -times integrated C -semigroup $\{U(t)\}_{t \geq 0}$, satisfying $\|U(t)\| \leq \varphi(t)e^{\omega t}$, and satisfies

$$\limsup_{\lambda \rightarrow \infty} \left\| \lambda(\lambda - A)^{-1} \right\| \leq M \quad (\lambda > \omega), \quad (2.30)$$

then

- (1) for $\lambda > \omega$, $x \in X$, $U(t)x = (\lambda - A_0)S_0(t)(\lambda - A)^{-1}x$,
- (2) for $x \in \overline{D(A)}$, $\lim_{t \rightarrow 0^+} U(t)x = 0$,
- (3) for $\lambda > \omega$, $x \in X$, $U(t)x = \lim_{\lambda \rightarrow \infty} \lambda S_0(t)(\lambda - A)^{-1}x$,
- (4) for $\lambda > \omega$, $x \in \overline{D(A)}$ if and only if $\lim_{\lambda \rightarrow \infty} \lambda^{n+1} \int_0^\infty e^{-\lambda t} U(t)x dt = Cx$,

where A_0 and $S_0(t)$ are the symbols mentioned in Theorem 2.3.

Proof. It follows from Theorems 2.3 and 2.4 that A subgenerates an $(n + 1)$ -times integrated C -semigroup $\{S(t)\}_{t \geq 0}$, which is continuously differentiable for $t > 0$ and satisfies (2.16) and (2.17).

Differentiating (2.16) with respect to t , we obtain

$$U(t)x = \frac{d}{dt}S(t)x = (\lambda - A_0)S_0(t)(\lambda - A)^{-1}x, \quad x \in X, \lambda > \omega. \quad (2.31)$$

This completes the proof of (1).

To show (2), for $x \in \overline{D(A)}$, we have

$$U(t)x = (\lambda - A_0)S_0(t)(\lambda - A)^{-1}x = S_0(t)x. \quad (2.32)$$

Letting $t \rightarrow 0^+$, we get

$$\lim_{t \rightarrow 0^+} U(t)x = 0, \quad x \in \overline{D(A)}. \quad (2.33)$$

To show (3), for $x \in X$, since $S(t)x \in C^1((0, +\infty), X)$, it follows from (2.17) that $\lim_{\lambda \rightarrow \infty} \lambda S_0(t)(\lambda - A)^{-1}x$ is continuous for $t > 0$, thus, we have

$$U(t)x = \frac{d}{dt}S(t)x = \lim_{\lambda \rightarrow \infty} \lambda S_0(t)(\lambda - A)^{-1}x, \quad t > 0. \quad (2.34)$$

Obviously, the equality above is true for $t = 0$.

Noting that

$$\limsup_{\lambda \rightarrow \infty} \|\lambda(\lambda - A)^{-1}\| \leq M \quad (\lambda > \omega), \quad (2.35)$$

we can deduce that $x \in \overline{D(A)}$ implies $\lim_{\lambda \rightarrow \infty} \lambda(\lambda - A)^{-1}Cx = Cx$, and from

$$(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} U(t)x dt, \quad (2.36)$$

assertion (4) is immediate if we note that $\lim_{\lambda \rightarrow \infty} \lambda(\lambda - A)^{-1}Cx = Cx$ implies $x \in \overline{D(A)}$. \square

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References

- [1] W. Arendt, "Vector-valued Laplace transforms and Cauchy problems," *Israel Journal of Mathematics*, vol. 59, no. 3, pp. 327–352, 1987.

- [2] A. Bobrowski, "On the generation of non-continuous semigroups," *Semigroup Forum*, vol. 54, no. 2, pp. 237–252, 1997.
- [3] Y.-C. Li and S.-Y. Shaw, "On local α -times integrated C -semigroups," *Abstract and Applied Analysis*, vol. 2007, Article ID 34890, 18 pages, 2007.
- [4] Y.-C. Li and S.-Y. Shaw, "On characterization and perturbation of local C -semigroups," *Proceedings of the American Mathematical Society*, vol. 135, no. 4, pp. 1097–1106, 2007.
- [5] J. Liang and T.-J. Xiao, "Integrated semigroups and higher order abstract equations," *Journal of Mathematical Analysis and Applications*, vol. 222, no. 1, pp. 110–125, 1998.
- [6] J. Liang and T.-J. Xiao, "Wellposedness results for certain classes of higher order abstract Cauchy problems connected with integrated semigroups," *Semigroup Forum*, vol. 56, no. 1, pp. 84–103, 1998.
- [7] J. Liang and T.-J. Xiao, "Norm continuity for $t > 0$ of linear operator families," *Chinese Science Bulletin*, vol. 43, no. 9, pp. 719–723, 1998.
- [8] K. Nagaoka, "Generation of the integrated semigroups by superelliptic differential operators," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1143–1154, 2008.
- [9] N. Tanaka, "Locally Lipschitz continuous integrated semigroups," *Studia Mathematica*, vol. 167, no. 1, pp. 1–16, 2005.
- [10] H. R. Thieme, "Integrated semigroups" and integrated solutions to abstract Cauchy problems," *Journal of Mathematical Analysis and Applications*, vol. 152, no. 2, pp. 416–447, 1990.
- [11] H. R. Thieme, "Differentiability of convolutions, integrated semigroups of bounded semi-variation, and the inhomogeneous Cauchy problem," *Journal of Evolution Equations*, vol. 8, no. 2, pp. 283–305, 2008.
- [12] T.-J. Xiao and J. Liang, "Integrated semigroups, cosine families and higher order abstract Cauchy problems," in *Functional Analysis in China*, vol. 356 of *Mathematics and Its Applications*, pp. 351–365, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
- [13] T.-J. Xiao and J. Liang, "Widder-Arendt theorem and integrated semigroups in locally convex space," *Science in China. Series A*, vol. 39, no. 11, pp. 1121–1130, 1996.
- [14] T.-J. Xiao and J. Liang, "Laplace transforms and integrated, regularized semigroups in locally convex spaces," *Journal of Functional Analysis*, vol. 148, no. 2, pp. 448–479, 1997.
- [15] T.-J. Xiao and J. Liang, *The Cauchy Problem for Higher-Order Abstract Differential Equations*, vol. 1701 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1998.
- [16] T.-J. Xiao and J. Liang, "Approximations of Laplace transforms and integrated semigroups," *Journal of Functional Analysis*, vol. 172, no. 1, pp. 202–220, 2000.
- [17] T.-J. Xiao and J. Liang, "Higher order abstract Cauchy problems: their existence and uniqueness families," *Journal of the London Mathematical Society*, vol. 67, no. 1, pp. 149–164, 2003.
- [18] T.-J. Xiao and J. Liang, "Second order differential operators with Feller-Wentzell type boundary conditions," *Journal of Functional Analysis*, vol. 254, no. 6, pp. 1467–1486, 2008.